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Multigravity and spacetime foam

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Abstract
We consider a multigravity approach to spacetime foam. As an application we give indications on the computation of the cosmological constant, considered as an eigenvalue of a Sturm–Liouville problem. A variational approach with Gaussian trial wavefunctionals is used as a method to study such a problem. We approximate the equation to one loop in a Schwarzschild background and a zeta function regularization is involved to handle with divergences. The regularization is closely related to the subtraction procedure appearing in the computation of the Casimir energy in a curved background. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation.

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Despite of its recent interest, the term '*multigravity*' is not completely new. Indeed, in the early seventies, some pioneering works appeared under the name '*strong gravity*' or '*f-g theory*' [1] as a tentative to describe a sector of hadronic physics where a massive spin-2 field (the *f-meson* with Planck mass $M_f \sim 1$ GeV) plays a dominant role. multigravity coincides with '*strong gravity*' or '*f-g theory*' when the number of gravitational fields involved is exactly 2 ('*bigravity*'). In this paper, we would like to use the multigravity idea as a model of space-time foam [2] to compute the cosmological constant. Such a computation has been done looking at the foam as a large N composition of Schwarzschild wormholes [3]. Nevertheless, the multigravity framework seems to be more appropriate for such a computation. We recall that there exists a very large discrepancy between the recent estimates on the cosmological constant, which approximately are of the order of 10^{-47} GeV^4, while a crude estimate of the zero point energy (ZPE) of some field of mass m with a cutoff at the Planck scale gives $E_{\mathrm{ZPE}} \approx 10^{71}$ GeV^4 with a difference of about 118 orders [4]. Let us see how to use multigravity, to approach such a problem. To this purpose we begin with the following action involving N massless gravitons without matter fields [5]:

$$S_0 = \sum_{i=1}^{N} S[g_i] = \sum_{i=1}^{N} \frac{1}{16\pi G_i} \int \mathrm{d}^4x \sqrt{-g_i}[R(g_i) - \Lambda_i], \tag{1}$$

where Λ_i and G_i are the cosmological constant and the related Newton constant corresponding to the ith universe, respectively. Generally speaking, the total action should be of the form

$$S_{\mathrm{tot}} = \sum_{i=1}^{N} S[g_i] + \lambda S_{\mathrm{int}}(g_1, g_2, \ldots, g_N). \tag{2}$$

When $\lambda \to 0$, the N world are non-interacting. This will be the context we are going to examine. In this way, the action S_0 describes a gas of gravitons. Consider for simplicity the case $N = 1$ and the related Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R^{(4)} + \Lambda_c g_{\mu\nu} = G_{\mu\nu} + \Lambda_c g_{\mu\nu} = 0, \tag{3}$$

where $G_{\mu\nu}$ is the Einstein tensor. If we introduce a time-like unit vector u^μ such that $u \cdot u = -1$, then

$$G_{\mu\nu} u^\mu u^\mu = \Lambda_c. \tag{4}$$

This is simply the Hamiltonian constraint written in terms of equation of motion, which is classical. However, the discrepancy between the observed cosmological constant and the computed one is in its quantum version, that could be estimate by the expectation value $\langle \Lambda_c \rangle$. Since

$$\frac{\sqrt{g}}{2\kappa} G_{\mu\nu} u^\mu u^\mu = \frac{\sqrt{g}}{2\kappa} R + \frac{2\kappa}{\sqrt{g}} \left(\frac{\pi^2}{2} - \pi^{\mu\nu} \pi_{\mu\nu} \right) = -\mathcal{H}, \tag{5}$$

where R is the scalar curvature in three dimensions, we can write

$$\frac{\langle \Lambda_c \rangle}{\kappa} = -\frac{1}{V} \left\langle \int_\Sigma \mathrm{d}^3x \mathcal{H} \right\rangle = -\frac{1}{V} \left\langle \int_\Sigma \mathrm{d}^3x \hat{\Lambda}_\Sigma \right\rangle, \tag{6}$$

where the last expression stands for

$$\frac{1}{V} \frac{\int \mathcal{D}[g_{ij}] \Psi^*[g_{ij}] \int_\Sigma \mathrm{d}^3x \mathcal{H} \Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}] \Psi^*[g_{ij}] \Psi[g_{ij}]} = \frac{1}{V} \frac{\langle \Psi | \int_\Sigma \mathrm{d}^3x \hat{\Lambda}_\Sigma | \Psi \rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}, \tag{7}$$

and where we have integrated over the hypersurface Σ, divided by its volume and functionally integrated over quantum fluctuation with the help of some trial wavefunctionals. Note that equation (7) can be derived starting with the Wheeler–DeWitt equation (WDW) [6] which represents invariance under *time* reparametrization. Equation (7) represents the Sturm–Liouville problem associated with the cosmological constant. The related boundary conditions are dictated by the choice of the trial wavefunctionals which, in our case are of the Gaussian type. Different types of wavefunctionals correspond to different boundary conditions. Extracting the TT tensor contribution from equation (7) approximated to second order in perturbation of the spatial part of the metric into a background term, $\bar{g}_{ij}$, and a perturbation, h_{ij}, we get

$$\hat{\Lambda}_\Sigma^\perp = \frac{1}{4V} \int_\Sigma \mathrm{d}^3x \sqrt{\bar{g}} G^{ijkl} \left[(2\kappa) K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(2\kappa)} (\triangle_2)^a_j K^\perp(x,x)_{iakl} \right]. \tag{8}$$

Here G^{ijkl} represents the inverse DeWitt metric and all indices run from one to three. The propagator $K^\perp(x,x)_{iakl}$ can be represented as

$$K^\perp(\vec{x}, \vec{y})_{iakl} := \sum_\tau \frac{h_{ia}^{(\tau)\perp}(\vec{x}) h_{kl}^{(\tau)\perp}(\vec{y})}{2\lambda(\tau)}, \tag{9}$$

where $h_{ia}^{(\tau)\perp}(\overrightarrow{x})$ are the eigenfunctions of $\triangle_2$, whose explicit expression for the massive case will be shown in the next section. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of equation (8). The expectation value of $\hat{\Lambda}_\Sigma^\perp$ is easily obtained by inserting the form of the propagator into equation (8) and minimizing with respect to the variational function $\lambda(\tau)$. Thus the total one-loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{4V}\sum_\tau\left[\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}\right]. \tag{10}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$, where ω_i are the eigenvalues of $\triangle_2$. If we fix our attention on some particular background, for example the Schwarzschild background, the spin-2 operator $\triangle_2$, simply becomes

The further step is the evaluation of equation (10). Its contribution to the spin-2 operator for the Schwarzschild metric will be

$$(\triangle_2 h^{TT})_i^j := -\triangle_S (h^{TT})_i^j + \frac{6}{r^2}\left(1-\frac{2MG}{r}\right)(h^{TT})_i^j + 2(Rh^{TT})_i^j. \tag{11}$$

$\triangle_S$ is the scalar curved Laplacian, whose form is

$$\triangle_S = \left(1-\frac{2MG}{r}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(\frac{2r-3MG}{r^2}\right)\frac{\mathrm{d}}{\mathrm{d}r} - \frac{L^2}{r^2} \tag{12}$$

and R_j^a is the mixed Ricci tensor whose components are

$$R_i^a = \left\{-\frac{2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3}\right\}. \tag{13}$$

This implies that the scalar curvature is traceless. We are therefore led to study the following eigenvalue equation:

$$(\triangle_2 h^{TT})_i^j = \omega^2 h_j^i, \tag{14}$$

where ω^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analysing the equation as modes of definite frequency, angular momentum and parity [7]. In particular, our choice for the three-dimensional gravitational perturbation is represented by its even-parity form

$$(h^{\mathrm{even}})_j^i(r,\vartheta,\phi) = \mathrm{diag}[H(r), K(r), L(r)]\, Y_{lm}(\vartheta,\phi). \tag{15}$$

Defining reduced fields and passing to the proper geodesic distance from the *throat* of the bridge, system (14) becomes

$$\begin{cases}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{l(l+1)}{r^2} + m_1^2(r)\right] f_1(x) = \omega_{1,l}^2 f_1(x) \\ \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{l(l+1)}{r^2} + m_2^2(r)\right] f_2(x) = \omega_{2,l}^2 f_2(x),\end{cases} \tag{16}$$

where we have defined $r \equiv r(x)$ and

$$\begin{cases} m_1^2(r) = U_1(r) = m_1^2(r,M) - m_2^2(r,M) \\ m_2^2(r) = U_2(r) = m_1^2(r,M) + m_2^2(r,M). \end{cases} \tag{17}$$

$m_1^2(r,M) \to 0$ when $r \to \infty$ or $r \to 2MG$ and $m_2^2(r,M) = 3MG/r^3$. Note that, while $m_2^2(r)$ is constant in sign, $m_1^2(r)$ is not. Indeed, for the critical value $\bar{r} = 5MG/2$, $m_1^2(\bar{r}) = m_g^2$ and in the range $(2MG, 5MG/2)$ for some values of m_g^2, $m_1^2(\bar{r})$ can be negative. It is interesting therefore concentrate in this range, where $m_1^2(r,M)$ vanishes when compared with $m_2^2(r,M)$. So, in a first approximation we can write

$$\begin{cases} m_1^2(r) \simeq -m_2^2(r_0,M) \\ m_2^2(r) \simeq +m_2^2(r_0,M), \end{cases} \tag{18}$$

where we have defined a parameter $r_0 > 2MG$ and $m_0^2(r_0,M) = 3MG/r_0^3$. The main reason for introducing a new parameter resides in the fluctuation of the horizon that forbids any kind of approach. It is now possible to explicitly evaluate equation (10) in terms of the effective mass. By adopting the WKB method used by 't Hooft in the brick wall problem [8], we arrive at the following relevant expression:

$$\rho_i(\varepsilon) = \frac{m_i^4(r)}{256\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m_i^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right], \tag{19}$$

$i = 1, 2$, where we have used the zeta function regularization method to compute the energy densities ρ_i and where we have introduced the additional mass parameter μ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. The energy density is renormalized via the absorption of the divergent part (in the limit $\varepsilon \to 0$) into the redefinition of the bare classical constant Λ:

$$\Lambda \to \Lambda_0 + \Lambda^{\text{div}} = \Lambda_0 + \frac{G}{32\pi\varepsilon}\left(m_1^4(r) + m_2^4(r)\right). \tag{20}$$

To remove the dependence on the arbitrary mass scale μ, it is appropriate to use the renormalization group equation. Therefore we impose that [9]

$$\frac{1}{8\pi G}\mu\frac{\partial\Lambda_0^{TT}(\mu)}{\partial\mu} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\rho_{eff}^{TT}(\mu,r), \tag{21}$$

where $\rho_{eff}^{TT}(\mu,r)$ is the renormalized energy density. Solving it we find that the renormalized constant Λ_0 should be treated as a running one in the sense that it varies provided that the scale μ is changing

$$\Lambda_0(\mu,r) = \Lambda_0(\mu_0,r) + \frac{G}{16\pi}\left(m_1^4(r) + m_2^4(r)\right)\ln\frac{\mu}{\mu_0}. \tag{22}$$

The final form for the cosmological constant is [10]

$$\frac{\Lambda_0(\mu_0,r_0)}{8\pi G} = -\frac{m_0^4(r_0,M)}{128\pi^2}\ln\left(\frac{m_0^2(r_0,M)\sqrt{e}}{4}\right) \tag{23}$$

which has a minimum for

$$\frac{m_0^2(r_0,M)\sqrt{e}}{4\mu_0^2} = \frac{1}{\sqrt{e}} \tag{24}$$

with

$$\frac{\Lambda_0(\mu_0,r)}{8\pi G} = -\frac{\mu_0^4}{16e^2\pi^2} \tag{25}$$

We can now discuss the multigravity gas. For each gravitational field introduce the following variables $(N,N_i)^{k)}$ and choose the gauge $N_i^{(k)} = 0, (k = 1\ldots N_w)$. Define the following domain $D_\Lambda^{(k)}$:

$$\left\{\Psi\,\middle|\,\left[\left[(2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}R\right]^{(k)}\Psi^{(k)}\left[g_{ij}^{(k)}\right] = -\frac{\sqrt{g^{(k)}}}{\kappa^{(k)}}\Lambda_c^{(k)}\Psi^{(k)}\left[g_{ij}^{(k)}\right]\right]\right\}, \tag{26}$$

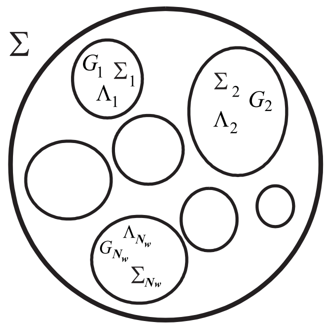

Figure 1. The space Σ composed by the non-overlapping spaces Σ_k.

and assume the following assumption: $\exists$ a covering of Σ such that

$$\Sigma = \bigcup_{k=1}^{N_w} \Sigma_k \qquad \Sigma_k \cap \Sigma_j = \emptyset \tag{27}$$

for $k \neq j$. Then equation (7) turns into

$$\frac{1}{V_{(k)}} \frac{\int \mathcal{D}\big[g_{ij}^{(k)}\big] \Psi^*_{(k)}\big[g_{ij}^{(k)}\big] \int_{\Sigma_k} \mathrm{d}^3x \, \hat{\Lambda}^{(k)}_{\Sigma_k} \Psi_{(k)}\big[g_{ij}^{(k)}\big]}{\int \mathcal{D}\big[g_{ij}^{(k)}\big] \Psi^*_{(k)}\big[g_{ij}^{(k)}\big] \Psi_{(k)}\big[g_{ij}^{(k)}\big]} = -\frac{\Lambda_{(k)}}{8\pi G_{(k)}}. \tag{28}$$

Each Σ_k has topology $S^2 \times R^1$. Therefore, the whole physical space Σ containing the energy density appears depicted as in the following picture: A further simplification comes from the assumption that the different Newton's constants are all equal. This leads to a model which is composed by N_w copies of the same world [10] and on each copy the procedure contained between equations (10) and (23) has to be repeated. Thus, the final evaluation of the '*global*' cosmological constant can be written as

$$\max\left\{\frac{1}{V_1}\int_{\Sigma_1} \mathrm{d}^3x \, \Lambda^{(1)}_{\Sigma_1}\right\} + \cdots + \max\left\{\frac{1}{V_{N_w}}\int_{\Sigma_{N_w}} \mathrm{d}^3x \, \Lambda^{(N_w)}_{\Sigma_{N_w}}\right\} = -\frac{\Lambda}{8\pi G}, \tag{29}$$

where $\Lambda^{(k)}_{\Sigma_k}$ is the eigenvalue obtained evaluating equation (28) on each Σ_k. The computation of the max is taken on each disjoint Σ_k. Note that in any case, the maximum of $\Lambda^{(k)}_{\Sigma_k}$ corresponds to the minimum of the energy density computed on the related hypersurface. It is interesting also to note that the whole procedure can be applied even in the case of a massive graviton [11] with a term of the form [12, 13]

$$S_m = \frac{m_g^2}{8\kappa} \int \mathrm{d}^4x \sqrt{-\hat{g}} \, [h^{ij} h_{ij}], \tag{30}$$

which is a particular sub-case of the Pauli–Fierz term [14].

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